

UNIVERSAL C^* -ALGEBRA OF REAL RANK ZERO

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ABSTRACT. It is well-known that every commutative separable unital C^* -algebra of real rank zero is a quotient of the C^* -algebra of all complex continuous functions defined on the Cantor cube. We prove a non-commutative version of this result by showing that the class of all separable unital C^* -algebras of real rank zero coincides with the class of quotients of a certain separable unital C^* -algebra of real-rank zero.

1. INTRODUCTION

It is a well-known observation (see, for instance, [5, Theorem 1.3.15]) that the Cantor cube $\{0, 1\}^\omega$ contains topological copy of any zero-dimensional metrizable compact space. By Gelfand's duality and by [2, Proposition 1.1], this means that every commutative separable unital C^* -algebra of real rank zero is a quotient of the C^* -algebra $C(\{0, 1\}^\omega)$. Our goal in the present note is to extend this result to the non-commutative case.

For a given class \mathcal{C} of separable unital C^* -algebras an element $Z \in \mathcal{C}$ is said to be universal (in \mathcal{C}) if every other element $X \in \mathcal{C}$ can be represented as the image of Z under a surjective unital $*$ -homomorphism. One particular method of obtaining universal elements for the class \mathcal{C} is to prove the existence of \mathcal{C} -invertible morphism $p: C^*(\mathbb{F}_\infty) \rightarrow Z$. Here $C^*(\mathbb{F}_\infty)$ stands for the group C^* -algebra of the free group on countable number of generators and the \mathcal{C} -invertibility of p means that for any unital $*$ -homomorphism $g: C^*(\mathbb{F}_\infty) \rightarrow X$, with $X \in \mathcal{C}$, there exists a unital $*$ -homomorphism $h: Z \rightarrow X$ such that $g = h \circ p$. It is easy to see that in such a situation Z is an universal element in the class \mathcal{C} . Indeed, every element X of \mathcal{C} (and, in general, every separable unital C^* -algebra) can be represented as the image of $C^*(\mathbb{F}_\infty)$ under a unital $*$ -homomorphism $g: C^*(\mathbb{F}_\infty) \rightarrow X$. By the \mathcal{C} -invertibility of p and by the fact that $X \in \mathcal{C}$, there exists a unital $*$ -homomorphism $h: Z \rightarrow X$ such that $g = h \circ p$. Since g is surjective it follows from the latter equality that h is surjective as well.

In the present note we consider the case $\mathcal{C} = \mathcal{RR}_0$, where \mathcal{RR}_0 denotes the class of all separable unital C^* -algebras of real rank zero, and prove that there

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indeed exists a \mathcal{RR}_0 -invertible morphism $p: C^*(\mathbb{F}_\infty) \rightarrow Z$ such that $Z \in \mathcal{RR}_0$ which, as noted, implies that Z is universal C^* -algebra in the class \mathcal{RR}_0 .

2. PRELIMINARIES

All C^* -algebras below are assumed to be unital and all $*$ -homomorphisms between unital C^* -algebras are also assumed to be unital. When we refer to a unital C^* -subalgebra of a unital C^* -algebra we implicitly assume that the inclusion is a unital $*$ -homomorphism. The set of all self-adjoint elements of a C^* -algebra X is denoted by X_{sa} . The density $d(X)$ of a C^* -algebra X is the minimal cardinality of dense subspaces (in a purely topological sense) of X . Thus $d(X) \leq \omega$ (ω denotes the first infinite cardinal number) means that X is separable.

2.1. Set-theoretical facts. For the reader's convenience we begin by presenting necessary set-theoretic facts. Their complete proofs can be found in [4].

Let A be a partially ordered *directed set* (i.e. for every two elements $\alpha, \beta \in A$ there exists an element $\gamma \in A$ such that $\gamma \geq \alpha$ and $\gamma \geq \beta$). We say that a subset $A_1 \subseteq A$ of A *majorates* another subset $A_2 \subseteq A$ of A if for each element $\alpha_2 \in A_2$ there exists an element $\alpha_1 \in A_1$ such that $\alpha_1 \geq \alpha_2$. A subset which majorates A is called *cofinal* in A . A subset of A is said to be a *chain* if every two elements of it are comparable. The symbol $\sup B$, where $B \subseteq A$, denotes the lower upper bound of B (if such an element exists in A). Let now τ be an infinite cardinal number. A subset B of A is said to be τ -*closed* in A if for each chain $C \subseteq B$, with $|C| \leq \tau$, we have $\sup C \in B$, whenever the element $\sup C$ exists in A . Finally, a directed set A is said to be τ -*complete* if for each chain C of elements of A with $|C| \leq \tau$, there exists an element $\sup C$ in A .

The standard example of a τ -complete set can be obtained as follows. For an arbitrary set A let $\exp A$ denote, as usual, the collection of all subsets of A . There is a natural partial order on $\exp A$: $A_1 \geq A_2$ if and only if $A_1 \supseteq A_2$. With this partial order $\exp A$ becomes a directed set. If we consider only those subsets of the set A which have cardinality $\leq \tau$, then the corresponding subcollection of $\exp A$, denoted by $\exp_\tau A$, serves as a basic example of a τ -complete set.

Proposition 2.1. *Let $\{A_t : t \in T\}$ be a collection of τ -closed and cofinal subsets of a τ -complete set A . If $|T| \leq \tau$, then the intersection $\cap \{A_t : t \in T\}$ is also cofinal (in particular, non-empty) and τ -closed in A .*

Corollary 2.2. *For each subset B , with $|B| \leq \tau$, of a τ -complete set A there exists an element $\gamma \in A$ such that $\gamma \geq \beta$ for each $\beta \in B$.*

Proposition 2.3. *Let A be a τ -complete set, $L \subseteq A^2$, and suppose the following three conditions are satisfied:*

Existence: *For each $\alpha \in A$ there exists $\beta \in A$ such that $(\alpha, \beta) \in L$.*

Majorantness: *If $(\alpha, \beta) \in L$ and $\gamma \geq \beta$, then $(\alpha, \gamma) \in L$.*

τ -closeness: Let $\{\alpha_t : t \in T\}$ be a chain in A with $|T| \leq \tau$. If $(\alpha_t, \beta) \in L$ for some $\beta \in A$ and each $t \in T$, then $(\alpha, \beta) \in L$ where $\alpha = \sup\{\alpha_t : t \in T\}$.

Then the set $\{\alpha \in A : (\alpha, \alpha) \in L\}$ is cofinal and τ -closed in A .

2.2. Direct C_τ^* -systems of C^* -algebras. Recall (see [8, Section 1.23] for details) that a direct system $\mathcal{S} = \{X_\alpha, i_\alpha^\beta, A\}$ of unital C^* -algebras consists of a partially ordered directed indexing set A , unital C^* -algebras X_α , $\alpha \in A$, and unital $*$ -homomorphisms $i_\alpha^\beta : X_\alpha \rightarrow X_\beta$, defined for each pair of indexes $\alpha, \beta \in A$ with $\alpha \leq \beta$, and satisfying the condition $i_\alpha^\gamma = i_\beta^\gamma \circ i_\alpha^\beta$ for each triple of indexes $\alpha, \beta, \gamma \in A$ with $\alpha \leq \beta \leq \gamma$. The (inductive) limit of the above direct system is a unital C^* -algebra which is denoted by $\varinjlim \mathcal{S}$. For each $\alpha \in A$ there exists a unital $*$ -homomorphism $i_\alpha : X_\alpha \rightarrow \varinjlim \mathcal{S}$ which will be called the α -th limit homomorphism of \mathcal{S} .

If A' is a directed subset of the indexing set A , then the subsystem $\{X_\alpha, i_\alpha^\beta, A'\}$ of \mathcal{S} is denoted $\mathcal{S}|_{A'}$.

Below in Section 3 we use the concept of the direct C_τ^* -system introduced in [3].

Definition 2.4. Let $\tau \geq \omega$ be a cardinal number. A direct system $\mathcal{S} = \{X_\alpha, i_\alpha^\beta, A\}$ of unital C^* -algebras X_α and unital $*$ -homomorphisms $i_\alpha^\beta : X_\alpha \rightarrow X_\beta$ is called a *direct C_τ^* -system* if the following conditions are satisfied:

- (a) A is a τ -complete set.
- (b) Density of X_α is at most τ (i.e. $d(X_\alpha) \leq \tau$), $\alpha \in A$.
- (c) The α -th limit homomorphism $i_\alpha : X_\alpha \rightarrow \varinjlim \mathcal{S}$ is an injective $*$ -homomorphism for each $\alpha \in A$.
- (d) If $B = \{\alpha_t : t \in T\}$ is a chain of elements of A with $|T| \leq \tau$ and $\alpha = \sup B$, then the limit homomorphism $\varinjlim \{i_{\alpha_t}^\alpha : t \in T\} : \varinjlim (\mathcal{S}|_B) \rightarrow X_\alpha$ is an isomorphism.

Proposition 2.5 (Proposition 3.2, [3]). *Let τ be an infinite cardinal number. Every unital C^* -algebra X can be represented as the limit of a direct C_τ^* -system $\mathcal{S}_X = \{X_\alpha, i_\alpha^\beta, A\}$ where the indexing set A coincides with $\exp_\tau Y$ for some (any) dense subset Y of X with $|Y| = d(X)$.*

Proof. If $d(X) \leq \tau$, then consider the direct C_τ^* -system $\mathcal{S}_X = \{X_\alpha, i_\alpha^\beta, \exp_\tau d(X)\}$, where $X_\alpha = X$ for each $\alpha \in \exp_\tau d(X)$ and $i_\alpha^\beta = \text{id}_X$ for each $\alpha, \beta \in \exp_\tau d(X)$ with $\alpha \leq \beta$.

If $d(X) > \tau$, then consider any subset Y of X such that $\text{cl}_X Y = X$ and $|Y| = d(X)$. Without loss of generality we may assume that Y contains the unit of X . Each $\alpha \in \exp_\tau d(X)$ can obviously be identified with a subset (denoted by the same letter α) of Y of cardinality $\leq \tau$. Let X_α be the smallest C^* -subalgebra of X containing α . If $\alpha, \beta \in \exp_\tau d(X)$ and $\alpha \leq \beta$, then $\alpha \subseteq \beta$ (as subsets of Y) and consequently $X_\alpha \subseteq X_\beta$. This inclusion map is denoted by

$i_\alpha^\beta: X_\alpha \rightarrow X_\beta$. It is easy to verify that the collection $\mathcal{S}_X = \{X_\alpha, i_\alpha^\beta, \exp_\tau d(X)\}$ is indeed a direct C_τ^* -system such that $\varinjlim \mathcal{S}_X = X$. \square

Lemma 2.6 (Lemma 3.3, [3]). *If $\mathcal{S}_X = \{X_\alpha, i_\alpha^\beta, A\}$ is a direct C_τ^* -system, then*

$$\varinjlim \mathcal{S}_X = \bigcup \{i_\alpha(X_\alpha) : \alpha \in A\}.$$

Proof. Clearly $\bigcup \{i_\alpha(X_\alpha) : \alpha \in A\}$ is dense in $\varinjlim \mathcal{S}_X$ (this fact remains true for arbitrary direct systems of C^* -algebras). Consequently, for any point $x \in \varinjlim \mathcal{S}_X$ there exists a sequence $\{x_n : n \in \omega\}$, consisting of elements from $\bigcup \{i_\alpha(X_\alpha) : \alpha \in A\}$, such that $x = \lim \{x_n : n \in \omega\}$. For each $n \in \omega$ choose an index $\alpha_n \in A$ such that $x_n \in i_{\alpha_n}(X_{\alpha_n})$. By Corollary 2.2, there exists an index $\alpha \in A$ such that $\alpha \geq \alpha_n$ for each $n \in \omega$. Since $i_{\alpha_n} = i_\alpha \circ i_{\alpha_n}^\alpha$, it follows that

$$x_n \in i_{\alpha_n}(X_{\alpha_n}) = i_\alpha(i_{\alpha_n}^\alpha(X_{\alpha_n})) \subseteq i_\alpha(X_\alpha) \quad \text{for each } n \in \omega.$$

Finally, since $i_\alpha(X_\alpha)$ is closed in $\varinjlim \mathcal{S}_X$, it follows that

$$x = \lim \{x_n : n \in \omega\} \in i_\alpha(X_\alpha).$$

\square

3. EXISTENCE OF AN UNIVERSAL SEPARABLE UNITAL C^* -ALGEBRA OF REAL RANK ZERO

The real rank of a unital C^* -algebra A , denoted by $RR(A)$, is defined as follows [2]. We say that $RR(A) \leq n$ if for each $(n+1)$ -tuple (x_1, \dots, x_{n+1}) of self-adjoint elements in A and every $\epsilon > 0$, there exists an $(n+1)$ -tuple (y_1, \dots, y_{n+1}) in A_{sa} such that $\sum y_k^2$ is invertible and

$$\left\| \sum_{k=1}^{n+1} (x_k - y_k)^2 \right\| < \epsilon.$$

Obviously unital C^* -algebras of real rank zero are defined as those in which every self-adjoint element can be arbitrarily closely approximated by self-adjoint invertible elements.

Proposition 3.1. *Let $\tau \geq \omega$ and X be an unital C^* -algebra. Then the following conditions are equivalent:*

1. $RR(X) = 0$.
2. X can be represented as the direct limit of a direct C_τ^* -system $\{X_\alpha, i_\alpha^\beta, A\}$ satisfying the following properties:
 - (a) The indexing set A is cofinal and τ -closed in the τ -complete set $\exp_\tau Y$ for some (any) dense subset Y of X with $|Y| = d(X)$.
 - (b) X_α is a C^* -subalgebra of X such that $RR(X_\alpha) = 0$, $\alpha \in A$.

Proof. The implication (2) \implies (1) follows from [2, Proposition 3.1].

In order to prove the implication (1) \implies (2) we first consider a direct C_τ^* -system $\mathcal{S}_X = \{X_\alpha, i_\alpha^\beta, A\}$ with properties indicated in Proposition 2.5. Next consider the following relation $L \subseteq A^2$:

$$L = \left\{ (\alpha, \beta) \in A^2 : \alpha \leq \beta \text{ and for each } \epsilon > 0 \text{ and for each } x \in (X_\alpha)_{sa} \text{ there exists an invertible } y \in (X_\beta)_{sa} \text{ such that } \|y - x\| < \epsilon \right\}.$$

Let us verify conditions of Proposition 2.3.

Existence. Let $\alpha \in A$ and $x \in (X_\alpha)_{sa}$. First we prove the following assertion.

(*) $_{(\alpha, x, \frac{1}{n})}$ There exist an index $\beta = \beta(\alpha, x, \frac{1}{n}) \in A$, $\beta \geq \alpha$ and an invertible element

$$y = y(\alpha, x, \frac{1}{n}) \in (X_\beta)_{sa} \text{ such that } \|y - x\| < \frac{1}{n}$$

Proof of () $_{(\alpha, x, \frac{1}{n})}$.* By (1), $RR(X) = 0$ and consequently there exists an

invertible element $y \in X_{sa}$ such that $\|y - x\| < \frac{1}{n}$. Since \mathcal{S}_X is a direct C_τ^* -system it follows from Lemma 2.6 that there exists an index $\alpha' \in A$ such that $y \in X_{\alpha'}$. Since A is a directed set there exists an index $\beta = \beta(\alpha, x, \frac{1}{n}) \in A$ such that $\alpha, \alpha' \leq \beta$. This obviously implies that $X_\beta \supseteq X_\alpha \cup X_{\alpha'} \supseteq X_\alpha \cup \{y\}$. This finishes the proof of (*) $_{(\alpha, x, \frac{1}{n})}$.

For a given element $x \in (X_\alpha)_{sa}$ consider indices $\beta(\alpha, x, \frac{1}{n}) \in A$, satisfying conditions (*) $_{(\alpha, x, \frac{1}{n})}$, $n = 1, 2, \dots$. By Corollary 2.2, there exists an element $\beta(\alpha, x) \in A$ such that $\beta(\alpha, x) \geq \beta(\alpha, x, \frac{1}{n})$ for each $n = 1, 2, \dots$. Obviously $X_\alpha \subseteq X_{\beta(\alpha, x)}$, $n = 1, 2, \dots$. Note also that

(*) $_{(\alpha, x)}$ For any $\epsilon > 0$ there exists an invertible element $y \in (X_{(\alpha, x)})_{sa}$ such that $\|y - x\| < \epsilon$.

Since \mathcal{S}_X is a direct C_τ^* -system it follows that $d(X_\alpha) \leq \tau$. Consequently there exists a subset $Y_\alpha \subseteq (X_\alpha)_{sa}$ such that $(X_\alpha)_{sa} = \text{cl}_{(X_\alpha)_{sa}} Y_\alpha$ and $|Y_\alpha| = d((X_\alpha)_{sa}) \leq d(X_\alpha) \leq \tau$. Let $Y_\alpha = \{y_\gamma : \gamma < \tau\}$. For each $\gamma < \tau$ consider an index $\beta(\alpha, y_\gamma) \in A$ satisfying condition (*) $_{(\alpha, y_\gamma)}$. Since A is a τ -complete set, we conclude by Corollary 2.2, there exists an index $\beta = \beta(\alpha) \in A$ such that $\beta \geq \beta(\alpha, y_\gamma)$ for each $\gamma \in \tau$.

We claim that $(\alpha, \beta) \in L$. Indeed, let $\epsilon > 0$ and $x \in (X_\alpha)_{sa}$. Since, by the above construction, $Y_\alpha \subseteq (X_\alpha)_{sa}$ is dense in $(X_\alpha)_{sa}$, there exists a self-adjoint element $y_\gamma \in Y_\alpha$ such that $\|y_\gamma - x\| \leq \frac{\epsilon}{2}$. By condition (*) $_{(\alpha, y_\gamma)}$ there exists an invertible element $y \in (X_{(\alpha, x)})_{sa}$ such that $\|y - y_\gamma\| < \frac{\epsilon}{2}$. Since $\beta(\alpha) \geq \beta(\alpha, y_\gamma)$, it follows that $X_{\beta(\alpha, y_\gamma)} \subseteq X_{\beta(\alpha)}$. This guarantees that $y \in (X_{\beta(\alpha)})_{sa}$. It only remains to note that $\|x - y\| < \epsilon$. Therefore $(\alpha, \beta) \in L$.

Majorantness. Let $(\alpha, \beta) \in L$, $\gamma \geq \beta$, $\epsilon > 0$ and $x \in (X_\alpha)_{sa}$. Since $(\alpha, \beta) \in L$, there exists an invertible element $y \in (X_\beta)_{sa}$ such that $\|y - x\| < \epsilon$. Since $\gamma \geq \beta$ it follows that $(X_\beta)_{sa} \subseteq (X_\gamma)_{sa}$ which shows that $y \in (X_\gamma)_{sa}$ and proves that $(\alpha, \gamma) \in L$.

τ -closeness. Suppose that $\{\alpha_t : t \in T\}$ is a chain of indices in A with $|T| \leq \tau$. Assume also that $(\alpha_t, \beta) \in L$ for each $t \in T$ where $\beta \in A$. Our goal is to show that $(\alpha, \beta) \in L$ where $\alpha = \sup\{\alpha_t : t \in T\}$. Let $\epsilon > 0$ and $x \in X_\alpha$. Since \mathcal{S}_X is a direct C_τ^* -system it follows that X_α is the direct limit of the direct system generated by C^* -subalgebras X_{α_t} , $t \in T$, and corresponding inclusion homomorphisms. Consequently there exist an index $t \in T$ and an element $x_t \in (X_{\alpha_t})_{sa}$ such that $\|x - x_t\| < \frac{\epsilon}{2}$. Since $(\alpha_t, \beta) \in L$ there exists an invertible element $y \in (X_\beta)_{sa}$ such that $\|x_t - y\| < \frac{\epsilon}{2}$. Clearly $\|x - y\| < \epsilon$. This shows that $(\alpha, \beta) \in L$.

We are now in position to apply Proposition 2.3 which guarantees that the set $A' = \{\alpha \in A : (\alpha, \alpha) \in L\}$ is cofinal and τ -closed in A . Note here that $(\alpha, \alpha) \in L$ precisely when for each $\epsilon > 0$ and for each element $x \in (X_\alpha)_{sa}$ there exists an invertible element $y \in (X_\alpha)_{sa}$ such that $\|x - y\| < \epsilon$. This means that the direct C_τ^* -system $\varinjlim \mathcal{S}'_X = \{X_\alpha, i_\alpha^\beta, A'\}$ consists of C^* -subalgebras of X of real rank zero. Clearly $\varinjlim \mathcal{S}'_X = X$. Proof is completed. \square

Corollary 3.2. *The following conditions are equivalent for any unital C^* -algebra X :*

1. $RR(X) = 0$.
2. X can be represented as the direct limit of a direct C_ω^* -system $\{X_\alpha, i_\alpha^\beta, A\}$ satisfying the following properties:
 - (a) The indexing set A is cofinal and ω -closed in the ω -complete set $\exp_\omega Y$ for some (any) dense subset Y of X with $|Y| = d(X)$.
 - (b) X_α is a separable unital C^* -subalgebra of X such that $RR(X_\alpha) = 0$, $\alpha \in A$.

Lemma 3.3. *Let $\{X_t : t \in T\}$ be an arbitrary collection of C^* -algebras and $RR(X_t) = 0$ for each $t \in T$. Then $RR\left(\prod\{X_t : t \in T\}\right) = 0$.*

Proof. This is an elementary exercise. Let $x = \{x_t : t \in T\} \in \prod\{X_t : t \in T\}$ be a self-adjoint element of the product $\prod\{X_t : t \in T\}$. This obviously means that $x_t = x_t^*$ for each $t \in T$. Let also $\epsilon > 0$. Since $RR(X_t) = 0$, $t \in T$, it follows that there exists a self-adjoint and invertible element $y_t \in X_t$ such that $\|x_t - y_t\| < \frac{\epsilon}{2}$, $t \in T$. Obviously $y = \{y_t : t \in T\} \in \prod\{X_t : t \in T\}$ and $\|x - y\| < \epsilon$ as required. \square

Next we construct a universal separable unital C^* -algebra Z of real rank zero. Universal in the sense that any other separable unital C^* -algebra with real rank zero is a quotient of Z . We note here that the group C^* -algebra $C^*(\mathbb{F}_\infty)$ of the free group on countable number of generators is certainly universal but its real rank is greater than zero. To see the latter assume the contrary, i.e. $RR(C^*(\mathbb{F}_\infty)) = 0$. By [6], $\text{cer}(C^*(\mathbb{F}_\infty)) < 1 + \epsilon$ which contradicts the fact [7, p.370] that $\text{cer}(C^*(\mathbb{F}_\infty)) = \infty$ (here cer stands for an exponential rank). Therefore $RR(C^*(\mathbb{F}_\infty)) > 0$.

Theorem 3.4. *The class \mathcal{RR}_0 of all separable unital C^* -algebras of real rank zero contains an universal element Z . More formally, there exists a \mathcal{RR}_0 -invertible unital $*$ -homomorphism $p: C^*(\mathbb{F}_\infty) \rightarrow Z$ where Z is a separable unital C^* -algebra such that $RR(Z) = 0$.*

Proof. Let $\mathcal{A} = \{f_t: C^*(\mathbb{F}_\infty) \rightarrow X_t, t \in T\}$ denote the set (see [1, Proposition 2.1]) of all unital $*$ -homomorphisms defined on $C^*(\mathbb{F}_\infty)$ such that $RR(X_t) = 0$. Next consider the product $\prod\{X_t: t \in T\}$. Since $RR(X_t) = 0$ for each $t \in T$ it follows from Lemma 3.3 that $RR\left(\prod\{X_t: t \in T\}\right) = 0$. The unital $*$ -homomorphisms $f_t, t \in T$, define the unique unital $*$ -homomorphism

$$f: C^*(\mathbb{F}_\infty) \rightarrow \prod\{X_t: t \in T\}$$

such that $\pi_t \circ f = f_t$ for each $t \in T$ (here $\pi_t: \prod\{X_t: t \in T\} \rightarrow X_t$ denotes the corresponding canonical projection $*$ -homomorphism). By Corollary 3.2, $\prod\{X_t: t \in T\}$ can be represented as the limit of the C_ω^* -system $\mathcal{S} = \{C_\alpha, i_\alpha^\beta, A\}$ such that C_α is a separable unital C^* -algebra of real rank zero for each $\alpha \in A$. Suppressing injective unital $*$ -homomorphisms $i_\alpha^\beta: C_\alpha \rightarrow C_\beta$ we for notational simplicity assume that C_α 's are unital C^* -subalgebras of $\prod\{X_t: t \in T\}$. Let $\{a_n: n \in \omega\}$ be a countable dense subset of $C^*(\mathbb{F}_\infty)$. By Lemma 2.6, for each $n \in \omega$ there exists an index $\alpha_n \in A$ such that $f(a_n) \in C_{\alpha_n}$. By Corollary 2.2, there exists an index $\alpha \in A$ such that $\alpha \geq \alpha_n$ for each $n \in \omega$. Then $f(a_n) \in C_{\alpha_n} \subseteq C_\alpha$ for each $n \in \omega$. This observation coupled with the continuity of f guarantees that

$$f(C^*(\mathbb{F}_\infty)) = f(\text{cl}(\{a_n: n \in \omega\})) \subseteq \text{cl}(f(\{a_n: n \in \omega\})) \subseteq \text{cl} C_\alpha = C_\alpha.$$

Let $Z = C_\alpha$ and p denote the unital $*$ -homomorphism f considered as the homomorphism of $C^*(\mathbb{F}_\infty)$ into Z . Note that $f = i \circ p$, where $i: Z = C_\alpha \hookrightarrow \prod\{X_t: t \in T\}$ stands for the inclusion.

By construction, $RR(Z) = 0$. Let us show that $p: C^*(\mathbb{F}_\infty) \rightarrow Z$ is \mathcal{RR}_0 -invertible in the sense of Introduction. In our situation for any unital $*$ -homomorphism $g: C^*(\mathbb{F}_\infty) \rightarrow X$, where X is a separable unital C^* -algebra of real rank zero, we need to establish the existence of a unital $*$ -homomorphism

$h: Z \rightarrow X$ such that $g = h \circ p$. Indeed, by definition of the set \mathcal{A} , we conclude that $g = f_t$ for some index $t \in T$ (in particular, $X = X_t$ for the same index $t \in T$). Next observe that $g = f_t = \pi_t \circ f = \pi_t \circ i \circ p$. This allows us to define the required unital $*$ -homomorphism $h: Z \rightarrow X$ as the composition $h = \pi_t \circ i$. This completes the proof of \mathcal{RR}_0 -invertibility of p and as a consequence (see Introduction) of the universality of Z . \square

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